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Higher order parallel surfaces in the Heisenberg group

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Abstract

We give a classification of k-parallel surfaces in the three-dimensional Heisenberg group. In particular, we prove that every k-parallel surface in the Heisenberg group is a vertical cylinder over a polynomial spiral of degree at most $k-1$.

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1. Introduction

Submanifolds with parallel second fundamental form are usually the first important class of submanifolds in a certain ambient Riemannian manifold to study, since they have the property of having the same extrinsic invariants at every point. Submanifolds with parallel second fundamental form, or briefly parallel submanifolds in real space forms are completely classified in [1] and independently in [14]. For a survey we refer to [12]. Higher order parallel submanifolds, i.e. submanifolds that satisfy $\overline{\nabla}^k h = 0$ for some positive integer k, were introduced more recently independently by F. Dillen in [5] and Ü. Lumiste in [11], for a survey we refer again to [12]. Since we focus on surfaces in this paper, we only quote the following result:

Theorem 1. *(See [5].) Let M^2 be a surface in $\mathbb{E}^3$. If M^2 is k-parallel, then it is an open part of a round sphere or of a cylinder on a polynomial spiral of degree at most $k-1$.*

A polynomial spiral of degree m is a curve in Euclidean plane whose curvature function is a polynomial function of the arc length of degree m, see for instance [16]. The classification of parallel hypersurfaces in real space forms can be found in [10], whereas for the classification of k-parallel hypersurfaces in real space forms we refer to [6] and [7]. In higher codimensions, the classification of parallel submanifolds in real space forms is found in [1].

Most results on parallel and higher order parallel submanifolds deal with submanifolds in real or complex space forms. On the other hand, in [3] parallel surfaces in the so-called Bianchi–Cartan–Vranceanu spaces, and in par-

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ticular in the Heisenberg group are classified. In the present paper we classify higher order parallel surfaces in the 3-dimensional Heisenberg group. In particular we prove the following theorem, which gives a partial affirmative answer to a conjecture stated in [3].

Main Theorem. *Every connected k-parallel surface in the Heisenberg group* Nil_3 *is an open part of a vertical cylinder on a plane curve, whose curvature function is a polynomial function of degree at most $k-1$ of the arc length.*

2. Parallel, semi-parallel and k-parallel hypersurfaces

Let $f : M^n \to \widetilde{M}^{n+1}$ be an isometric immersion of Riemannian manifolds. Denoting by N a unit normal vector field on the hypersurface and by ∇ and $\widetilde{\nabla}$ the Levi Civita connections of M^n and $\widetilde{M}^{n+1}$ respectively, we define for all $X, Y \in T_pM^n$, $p \in M^n$, the shape operator S by $SX = -\widetilde{\nabla}_X N$ and the second fundamental form h by $h(X,Y) = \langle SX, Y\rangle = \langle X, SY\rangle$.

The covariant derivative of h is defined by

$$(\nabla h)(X,Y,Z) = X\big[h(Y,Z)\big] - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for all $X, Y, Z \in T_pM^n$. If R is the curvature tensor of M^n, we also define

$$(R\cdot h)(X,Y,Z_1,Z_2) = -h\big(R(X,Y)Z_1, Z_2\big) - h\big(Z_1, R(X,Y)Z_2\big)$$

for all $X, Y, Z_1, Z_2 \in T_pM^n$. If $\nabla h = 0$, we say that M^n has parallel second fundamental form or, for short, that it is a *parallel* hypersurface. If $R\cdot h = 0$, we say that M^n is a *semi-parallel* hypersurface.

For any integer $k \geqslant 2$, we define recursively

$$\begin{aligned}\big(\nabla^k h\big)(X_1,\ldots,X_k,Y,Z) &= X_1\big[\big(\nabla^{k-1}h\big)(X_2,\ldots,X_k,Y,Z)\big]\\ &\quad - \big(\nabla^{k-1}h\big)(\nabla_{X_1}X_2,\ldots,X_k,Y,Z) - \cdots - \big(\nabla^{k-1}h\big)(X_2,\ldots,X_k,Y,\nabla_{X_1}Z)\end{aligned}$$

for $X_1,\ldots,X_k,Y,Z \in T_pM^n$, and

$$\begin{aligned}\big(R^k\cdot h\big)(X_1,Y_1,\ldots,X_k,Y_k,Z_1,Z_2) &= -\big(R^{k-1}\cdot h\big)\big(R(X_1,Y_1)X_2,Y_2,\ldots,X_k,Y_k,Z_1,Z_2\big) - \cdots\\ &\quad - \big(R^{k-1}\cdot h\big)\big(X_2,Y_2,\ldots,X_k,Y_k,Z_1,R(X_1,Y_1)Z_2\big)\end{aligned}$$

for $X_1,Y_1,\ldots,X_k,Y_k,Z_1,Z_2 \in T_pM^n$. We call a hypersurface satisfying $\nabla^k h = 0$ a *k-parallel* hypersurface. With slight modifications, all these notions can also be defined for submanifolds with arbitrary codimension.

3. The geometry of the Heisenberg group

The Heisenberg group Nil_3 is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined as

$$(x,y,z)*(\bar{x},\bar{y},\bar{z}) = \left(x+\bar{x},\ y+\bar{y},\ z+\bar{z}+\frac{x\bar{y}}{2}-\frac{\bar{x}y}{2}\right).$$

The identity of the group is $(0,0,0)$ and the inverse of (x,y,z) is given by $(-x,-y,-z)$. The left-invariant metric on Nil_3 is

$$ds^2 = dx^2 + dy^2 + \left(dz + \frac{y}{2}dx - \frac{x}{2}dy\right)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{e_1 = \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}\right\}.$$

The characterising properties of this algebra are the following commutation relations:

$$[e_1,e_2] = e_3; \qquad [e_2,e_3] = 0; \qquad [e_3,e_1] = 0.$$

The Levi Civita connection of Nil_3 is then given by

$$\begin{aligned}
&\widetilde{\nabla}_{e_1} e_1 = 0; \qquad \widetilde{\nabla}_{e_1} e_2 = \frac{1}{2} e_3; \qquad \widetilde{\nabla}_{e_1} e_3 = -\frac{1}{2} e_2;\\
&\widetilde{\nabla}_{e_2} e_1 = -\frac{1}{2} e_3; \qquad \widetilde{\nabla}_{e_2} e_2 = 0; \qquad \widetilde{\nabla}_{e_2} e_3 = \frac{1}{2} e_1;\\
&\widetilde{\nabla}_{e_3} e_1 = -\frac{1}{2} e_2; \qquad \widetilde{\nabla}_{e_3} e_2 = \frac{1}{2} e_1; \qquad \widetilde{\nabla}_{e_3} e_3 = 0.
\end{aligned} \tag{1}$$

Remark that $\widetilde{\nabla}_X e_3 = \frac{1}{2}(X \times e_3)$ for every $X \in T(\mathrm{Nil}_3)$, where the cross product is defined as a bilinear operation, satisfying $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = e_2$. The equations in (1) yield the following expression for the curvature tensor of Nil_3:

$$\begin{aligned}
\widetilde{R}(X,Y)Z = {}& \frac{3}{4}\big(\langle X,Z\rangle Y - \langle Y,Z\rangle X\big)\\
&+ \langle Y,e_3\rangle\langle Z,e_3\rangle X - \langle X,e_3\rangle\langle Z,e_3\rangle Y + \langle X,e_3\rangle\langle Y,Z\rangle e_3 - \langle Y,e_3\rangle\langle X,Z\rangle e_3
\end{aligned}$$

for $p \in \mathrm{Nil}_3$ and $X, Y, Z \in T_p(\mathrm{Nil}_3)$.

One of the key elements in the geometry of the Heisenberg group is the fact that the mapping

$$\pi : \mathrm{Nil}_3 \to \mathbb{E}^2 : (x,y,z) \mapsto (x,y)$$

is a Riemannian submersion, whose fibres are the integral curves of $e_3 = \frac{\partial}{\partial z}$. By a *vertical cylinder* we will denote the inverse image $\pi^{-1}\gamma$ of a curve γ in $\mathbb{E}^2$.

Let M^2 be an oriented surface in Nil_3 with unit normal N and shape operator S. We denote by θ the angle between e_3 and N and by T the projection of e_3 on the tangent space of M^2. The equations of Gauss and Codazzi become respectively

$$\begin{aligned}
R(X,Y)Z = {}& \frac{3}{4}\big(\langle X,Z\rangle Y - \langle Y,Z\rangle X\big) + \langle Y,T\rangle\langle Z,T\rangle X - \langle X,T\rangle\langle Z,T\rangle Y\\
&+ \langle X,T\rangle\langle Y,Z\rangle T - \langle Y,T\rangle\langle X,Z\rangle T + \langle SY,Z\rangle SX - \langle SX,Z\rangle SY
\end{aligned} \tag{2}$$

and

$$\nabla_X SY - \nabla_Y SX - S[X,Y] = \cos\theta\big(\langle X,T\rangle Y - \langle Y,T\rangle X\big) \tag{3}$$

for $p \in M^2$ and $X, Y, Z \in T_pM^2$. From (2) it follows that the Gaussian curvature of M^2 satisfies

$$K = \det S + \frac{1}{4} - \cos^2\theta. \tag{4}$$

Finally, we remark that the following structure equations hold for $p \in M^2$ and $X \in T_pM^2$:

$$\nabla_X T = \cos\theta\left(SX - \frac{1}{2}JX\right); \tag{5}$$

$$X[\cos\theta] = -\left\langle SX - \frac{1}{2}JX, T\right\rangle; \tag{6}$$

where J denotes the rotation over $\frac{\pi}{2}$ in T_pM^2. These equations can be verified straightforwardly by comparing the tangential and normal components of both sides of the equality $\widetilde{\nabla}_X(T + \cos\theta N) = \frac{1}{2}(X \times (T + \cos\theta N))$.

A surface in the Heisenberg group is completely determined by the metric and the entities T, S and θ. This is formulated in the following theorem taken from [4]; in fact in [4] a more general version is proved.

Theorem 2. *(See [4].) Let M^2 be a simply connected, oriented Riemannian surface. Let J denote the rotation over $\frac{\pi}{2}$ in TM^2 and S a field of symmetric operators on TM^2. Finally, let T be a vector field on M^2 and let $\cos\theta$ be a differentiable function, satisfying $\langle T,T\rangle + \cos^2\theta = 1$. Then there exists an isometric immersion of M^2 in Nil_3 with unit normal N, such that (after the appropriate identifications) S is the shape operator and $e_3 = T + \cos\theta N$ if and only if the Eqs. (2), (3), (5) and (6) are satisfied. In this case the immersion is moreover unique up to a global isometry of Nil_3, preserving both the orientations of the base space $\mathbb{E}^2$ and the fibres of π.*

We also quote the following theorem from [3].

Theorem 3. *(See [3].) The only parallel surfaces in the Heisenberg group* Nil_3 *are open parts of vertical planes and vertical round cylinders.*

Here, a vertical plane is the inverse image of a straight line under π and a vertical cylinder is the inverse image of a circle under π. The proof of this theorem, given in [3], uses the fact that if $\nabla h = 0$, the left hand side of Eq. (3) is zero. This technique cannot be straightforwardly generalized to classify k-parallel surfaces in Nil_3.

4. k-parallel versus semi-parallel surfaces

The following result holds for a surface M^2 immersed in an arbitrary three-dimensional Riemannian manifold, and is in fact a generalization of Lemma 2.2 in [5]. Since the proof is completely similar, we omit it.

Lemma 1. *Let $p \in M^2$ and let $\{E_1, E_2\}$ be an orthonormal basis for T_pM^2, such that $SE_i = \lambda_i E_i$. Denote by K the Gaussian curvature of M^2 at p. Then for $k \geqslant 0$:*

(i) $(R^{2k+1} \cdot h)(E_1, E_2, \dots, E_1, E_2, E_1, E_1) = 0$;
(ii) $(R^{2k+1} \cdot h)(E_1, E_2, \dots, E_1, E_2, E_1, E_2) = (-1)^{k+1}2^{2k}(\lambda_1 - \lambda_2)K^{2k+1}$;

and for $k > 0$:

(iii) $(R^{2k} \cdot h)(E_1, E_2, \dots, E_1, E_2, E_1, E_1) = (-1)^k 2^{2k-1}(\lambda_1 - \lambda_2)K^{2k}$;
(iv) $(R^{2k} \cdot h)(E_1, E_2, \dots, E_1, E_2, E_1, E_2) = 0$.

We can now prove the following:

Lemma 2. *A k-parallel surface immersed in a three-dimensional Riemannian manifold is semi-parallel, or equivalently, it is flat or totally umbilical.*

Proof. Consider a k-parallel surface in a three-dimensional Riemannian manifold. From Lemma 2.1 in [5] it follows that $R^{\lceil \frac{k}{2} \rceil} \cdot h = 0$. The proof of this lemma is based on the Ricci-identity. But from Lemma 1 above, we now see that $(\lambda_1 - \lambda_2)K = 0$, which implies the statement. $\square$

Since the Heisenberg group does not admit any totally umbilical surfaces, see for example [13], it follows that all k-parallel surfaces in Nil_3 are flat.

Example 1. Consider a vertical cylinder in Nil_3 and take an orthonormal frame on it of the form $\{E_1 = ae_1 + be_2, E_2 = e_3\}$, with $a^2 + b^2 = 1$. Then the vectorfield $N = E_1 \times E_2 = be_1 - ae_2$ is a unit normal and one can verify

$$\widetilde{\nabla}_{E_1} N = \big(aE_1[b] - bE_1[a]\big)E_1 - \frac{1}{2}E_2;$$
$$\widetilde{\nabla}_{E_2} N = \left(aE_2[b] - bE_2[a] - \frac{1}{2}\right)E_1;$$

from which

$$S = \begin{pmatrix} -aE_1[b] + bE_1[a] & -aE_2[b] + bE_2[a] + \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} -aE_1[b] + bE_1[a] & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

the last equality due to the symmetry of S. Remark that from this symmetry we have $aE_2[b] = bE_2[a]$, which, together with $a^2 + b^2 = 1$, yields that a and b are constant along the fibres of π. This will be used in the next section. From

Eq. (4), we now get that

$$K = \det S + \frac{1}{4} - \cos^2\theta = -\frac{1}{4} + \frac{1}{4} - \cos^2\frac{\pi}{2} = 0,$$

proving that the surface is flat and thus semi-parallel. Which vertical cylinders are moreover k-parallel will become clear in the next section.

Example 2. Let U be an open part of $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ a differentiable function. Define the surface M^2_f in Nil_3 as

$$M^2_f = \{(x, y, f(x, y)) \mid x, y \in U\}.$$

Then the following formula for the Gaussian curvature was proven in [2]:

$$\begin{aligned} W^4K = {} & W^2\left(f_{xy}^2 - f_{xx}f_{yy} - \frac{1}{4}\right) - (1+q^2)\left(\left(f_{xy} + \frac{1}{2}\right)^2 - f_{xx}f_{yy}\right) \\ & - (1+p^2)\left(\left(f_{xy} - \frac{1}{2}\right)^2 - f_{xx}f_{yy}\right) + pq(f_{yy} - f_{xx}), \end{aligned} \tag{7}$$

where the indices denote partial derivatives and $p = f_x + \frac{y}{2}$, $q = f_y - \frac{x}{2}$, $W = \sqrt{1+p^2+q^2}$. Putting $K = 0$, Eq. (7) becomes a partial differential equation for f. We will solve it in the special cases that

$$(x, y, f(x, y)) = (x, 0, \alpha(x)) * (0, y, \beta(y)) \tag{8}$$

or

$$(x, y, f(x, y)) = (0, y, \beta(y)) * (x, 0, \alpha(x)) \tag{9}$$

and moreover $\alpha(x) = 0$ or $\beta(y) = 0$. Remark that in some sense, we investigate which "cylinders" on curves in the surfaces $x = 0$ and $y = 0$ are flat. This notion of a cylinder in Nil_3 was introduced in [8] and flat ones were studied independently in [9]. Recall from the previous example that all cylinders on curves in the surface $z = 0$, $(x, \alpha(x), 0) * (0, 0, z) = (0, 0, z) * (x, \alpha(x), 0)$ are flat. In the case (8) we have $f(x, y) = \alpha(x) + \beta(y) + \frac{xy}{2}$ and (7) becomes for $K = 0$

$$1 + \beta'(y)^2 - \alpha''(x)\beta''(y) - \beta'(y)(\alpha'(x) + y)(\beta''(y) - \alpha''(x)) = 0.$$

For $\alpha = 0$ we get $1 + \beta'(y)^2 - y\beta'(y)\beta''(y) = 0$, with solutions

$$\beta(y) = \pm\frac{1}{2}\left(y\sqrt{A^2y^2 - 1} - \frac{1}{A}\ln\left|Ay + \sqrt{A^2y^2 - 1}\right|\right) + B.$$

For $\beta = 0$, the equation has no solutions. In the case (9) we have $f(x, y) = \alpha(x) + \beta(y) - \frac{xy}{2}$ and we get

$$1 + \alpha'(x)^2 - \alpha''(x)\beta''(y) - \alpha'(x)(\beta'(y) - x)(\beta''(y) - \alpha''(x)) = 0.$$

For $\alpha = 0$, there are no solutions, for $\beta = 0$, we find $1 + \alpha'(x)^2 - x\alpha'(x)\alpha''(x) = 0$, with as solutions again

$$\alpha(x) = \pm\frac{1}{2}\left(x\sqrt{A^2x^2 - 1} - \frac{1}{A}\ln\left|Ax + \sqrt{A^2x^2 - 1}\right|\right) + B.$$

In this way we constructed again some examples of semi-parallel surfaces in Nil_3.

5. Classification of the k-parallel surfaces in Nil_3

In this section we consider a k-parallel surface M^2 immersed in Nil_3. As we know from the remark after Lemma 2, M^2 must be flat and hence every $p \in M^2$ has an open neighbourhood U, which is isometric to an open part of $\mathbb{E}^2$. Denote by (u, v) the Euclidean coordinates on U. Suppose $T = T_1\frac{\partial}{\partial u} + T_2\frac{\partial}{\partial v}$ and $S = (S_{ij})_{1\leqslant i,j\leqslant 2}$ with respect to the orthonormal basis $\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$. We consider S_{11}, S_{12}, S_{22}, $\cos\theta$, T_1 and T_2 as functions of the Euclidean coordinates (u, v) on U.

Lemma 3. *The functions S_{11}, S_{12}, S_{22}, $\cos\theta$, T_1 and T_2 satisfy the following system of equations*:

$$T_1^2 + T_2^2 + \cos^2\theta = 1; \tag{10}$$

$$S_{11}S_{22} - S_{12}^2 + \frac{1}{4} - \cos^2\theta = 0; \tag{11}$$

$$\frac{\partial S_{12}}{\partial u} - \frac{\partial S_{11}}{\partial v} = -T_2\cos\theta; \tag{12a}$$

$$\frac{\partial S_{22}}{\partial u} - \frac{\partial S_{12}}{\partial v} = T_1\cos\theta; \tag{12b}$$

$$\frac{\partial T_1}{\partial u} = S_{11}\cos\theta; \tag{13a}$$

$$\frac{\partial T_1}{\partial v} = \left(S_{12} + \frac{1}{2}\right)\cos\theta; \tag{13b}$$

$$\frac{\partial T_2}{\partial u} = \left(S_{12} - \frac{1}{2}\right)\cos\theta; \tag{13c}$$

$$\frac{\partial T_2}{\partial v} = S_{22}\cos\theta; \tag{13d}$$

$$\frac{\partial \cos\theta}{\partial u} = -S_{11}T_1 - S_{12}T_2 + \frac{1}{2}T_2; \tag{14a}$$

$$\frac{\partial \cos\theta}{\partial v} = -S_{12}T_1 - S_{22}T_2 - \frac{1}{2}T_1. \tag{14b}$$

Proof. Eq. (10) follows immediately from the definitions of T and θ. Eq. (11) expresses Gauss' equation (4), while the equations (12) express the equation of Codazzi (3). The equations in (13) and (14) are nothing but the structure equations (5) and (6). □

Remark that Eqs. (14) can be seen as integrability conditions for the system of differential equations in (12) and (13). We can now prove our Main Theorem.

Proof of the Main Theorem. From $\nabla^k h = 0$, it follows that the functions S_{ij} are polynomials of degree at most $k-1$ in the Euclidean coordinates u and v. From (11) and (12) we get that also the functions $\cos^2\theta$, $T_1\cos\theta$ and $T_2\cos\theta$ are polynomials. By multiplying both sides of the equations in (13) with either T_1 or T_2, we see that the partial derivatives of the functions T_1^2 and T_2^2, and hence the functions themselves, are also polynomials.

Step 1: *θ is a constant*

Suppose first that none of the functions S_{11}, S_{12}, S_{22}, $\cos\theta$, T_1 and T_2 is zero. Put $\deg_u S_{11} = n_1$, $\deg_u S_{12} = n_2$, $\deg_u S_{22} = n_3$ and $\deg_v S_{11} = m_1$, $\deg_v S_{12} = m_2$, $\deg_v S_{22} = m_3$, with $n_i, m_i \in \mathbb{N}$.

First we prove that $n_2 - 1 \leqslant n_1$. Suppose that this would not be the case, so that $n_2 - 1 > n_1$. Then (12a) implies that $\deg_u(T_2\cos\theta) = n_2 - 1$, and (13c) implies that $\deg_u(\frac{\partial T_2^2}{\partial u}) = 2n_2 - 1$. (For both conclusions, we used that S_{12} is not independent of u, which follows from $n_2 > n_1 + 1 \geqslant 1$.) We conclude that $\deg_u(T_2^2) = 2n_2$ and $\deg_u(T_2^2\cos^2\theta) = 2n_2 - 2$, which is a contradiction.

We now prove that $n_3 - 1 \leqslant n_2$. Suppose again that this would not be the case, so that $n_3 - 1 > n_2$. Then from (12b), $\deg_u(T_1\cos\theta) = n_3 - 1$ and from (13a), $\deg_u(\frac{\partial T_1^2}{\partial u}) = n_1 + n_3 - 1$. Thus, $\deg_u(T_1^2) = n_1 + n_3$ and $\deg_u(T_1^2\cos^2\theta) = 2n_3 - 2$, from which $\deg_u(\cos^2\theta) = n_3 - n_1 - 2$. We may suppose that $\cos\theta$ is not a constant, in particular that it is not constantly $\frac{1}{4}$. Then it follows from (11) that also $\deg_u(S_{11}S_{22} - S_{12}^2) = n_3 - n_1 - 2$. But this means that we have to be necessarily in one of the following cases: $n_1 + n_3 = n_3 - n_1 - 2$, $2n_2 = n_3 - n_1 - 2$ or $n_1 + n_3 = 2n_2$. We see immediately that the first case is impossible. The last one is also impossible: from our assumption it follows that $n_3 \geqslant n_2 + 2$ and above we proved that $n_1 \geqslant n_2 - 1$, so $n_1 + n_3 \geqslant 2n_2 + 1$. Hence we get that $2n_2 = n_3 - n_1 - 2$ and $\deg_u(S_{11}S_{22} - S_{12}^2) = 2n_2$. So $n_1 + n_3 \leqslant 2n_2 = n_3 - n_1 - 2$, thus $2n_1 \leqslant -2$, which is again a contradiction.

Next, we prove that $n_2 = n_1 + 1$ and that $\cos\theta$ is independent of u. From (12b), $\deg_u(T_1\cos\theta) = \ell \leqslant n_2$, where we used that $T_1\cos\theta \neq 0$ and $n_3 - 1 \leqslant n_2$. Then from (13a) we get $\deg_u(\frac{\partial T_1^2}{\partial u}) = n_1 + \ell$, thus $\deg_u(T_1^2) = n_1 + \ell + 1$ and $\deg_u(T_1^2\cos^2\theta) = 2\ell$, such that $2\ell \geqslant n_1 + \ell + 1$ and $\ell \geqslant n_1 + 1 \geqslant n_2$. Because we also had $\ell \leqslant n_2$, this yields that $\ell = n_1 + 1 = n_2$. But then $\deg_u(\cos^2\theta) = 2\ell - (n_1 + \ell + 1) = n_2 - n_1 - 1 = 0$, such that $\cos\theta$ is independent of u, as stated.

Remark that $\cos\theta$ is also independent of v. Indeed, in an analogous way as above, we can prove successively that $m_2 - 1 \leqslant m_3$, $m_1 - 1 \leqslant m_2$, $m_2 = m_3 + 1$ and $\cos\theta$ is independent of v. In the proof we will use that $T_2\cos\theta \neq 0$.

To finish the first step of the proof, we prove that if one of the functions S_{11}, S_{12}, S_{22}, $\cos\theta$, T_1 or T_2 is constantly zero, we have that $\cos\theta = 0$.

- If $S_{11} = 0$, we get from (11) that $-S_{12}^2 + \frac{1}{4} - \cos^2\theta = 0$. Differentiating this equality with respect to u, we find, using (12a) and (14a), that $T_2\cos\theta(2S_{12} - \frac{1}{2}) = 0$. If $T_2 = 0$, we refer to one of the following cases, if $S_{12} = \frac{1}{4}$, we get from (12a) that either $T_2 = 0$ or that $\cos\theta = 0$.
- If $S_{22} = 0$, we analogously differentiate (12b) with respect to v.
- If $S_{12} = 0$, we may assume that $S_{11} \neq 0$ and $S_{22} \neq 0$. From (12b), $\deg_u(T_1\cos\theta) = n_3 - 1$, unless S_{22} is independent of u, but in that case $T_1\cos\theta = 0$ and we refer to one of the other cases. From (13a), we get $\deg_u(\frac{\partial T_1^2}{\partial u}) = n_1 + n_3 - 1$, thus $\deg_u(T_1^2) = n_1 + n_3$ and $\deg_u(T_1^2\cos^2\theta) = 2n_3 - 2$, from which $\deg_u(\cos^2\theta) = n_3 - n_1 - 2 \geqslant 0$. On the other hand, (11) implies that $\deg_u(\cos^2\theta) = n_1 + n_3$, (unless $S_{11}S_{22} = -\frac{1}{4}$, but in this case it follows immediately that $\cos\theta = 0$). This would imply that $n_3 - n_1 - 2 = n_1 + n_3$, or equivalently, that $n_1 = -1$, a contradiction.
- If $T_1 = 0$, we get from (13a) and (13b) that $S_{11}\cos\theta = (S_{12} + \frac{1}{2})\cos\theta = 0$. If $S_{11} = S_{12} + \frac{1}{2} = 0$, then from (11) we see that $\cos\theta = 0$.
- If $T_2 = 0$ the argument is analogous to the one above.

Step 2: *M^2 is an open part of a vertical cylinder*

Because θ is a constant, it follows from (12a) and (12b) that the functions T_1 and T_2 are polynomial functions in u and v. Since T_1 and T_2 satisfy $T_1^2 + T_2^2 = 1 - \cos^2\theta$ and θ is a constant, they have to be constant. Then (13b) and (13c) imply that $(S_{12} + \frac{1}{2})\cos\theta = (S_{12} - \frac{1}{2})\cos\theta = 0$, such that $\cos\theta = 0$ and thus $\theta = \frac{\pi}{2}$. Hence the surface is an open part of a vertical cylinder.

Step 3: *Assertion about the curvature of the base curve*

Take E_1 and E_2 as in Example 1. One can verify that $\nabla_{E_i}E_j = 0$, so we can choose Euclidean coordinates (u, v) such that $E_1 = \frac{\partial}{\partial u}$, $E_2 = \frac{\partial}{\partial v}$. Remark that the v-coordinate coincides with the z-coordinate of Nil_3. As we remarked before, a and b will now only depend on u and we write a' and b' for the derivatives with respect to u. The base curve $\gamma(u)$ satisfies $\gamma' = \pi_* E_1 = (a, b)$, such that u is an arc length parameter. We compute

$$\kappa_\gamma = \frac{ab' - a'b}{(a^2 + b^2)^{\frac{3}{2}}} = ab' - a'b = -S_{11}.$$

Looking at the expression for S, we see that the surface is k-parallel if and only if S_{11} is a polynomial of degree at most $k - 1$ in u and v. This is equivalent to κ_γ being a polynomial of degree at most $k - 1$ of u. $\square$

Remark 1. In the mean time, the second named author used similar techniques to extend the Main Theorem to a classification of higher order parallel surfaces in 3-dimensional Bianchi–Cartan–Vranceanu spaces, giving an affirmative answer to a conjecture formulated in [3] (see [15]). An important difference with the situation in the Heisenberg group is that one has to take into account possible totally umbilical surfaces. In [15], a full classification of totally umbilical surfaces in these spaces is given.

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